Mixing of three neutrinos in matter

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Abstract

Explicit analytical expression is derived for mixing matrix of three neutrinos in matter using a set of direct vacuum physical parameters. Results are presented in simple, symmetrical form. The physical contents are more clear than using of traditional mixing angles. There is no problem about mixing orders.

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I. INTRODUCTION

The deficit of the solar neutrinos leads to the studies of neutrino oscillations in matter [1]. Many important results have been obtained for two flavors. An important fact, discovered more than ten years ago, is that there is resonance of the mixing angle in matter due to the interaction of charged current [2]. Because of the increase of mixing parameters the situation is more complex in three flavors than two [3].

The supposition of small vacuum mixing angles between three flavor neutrinos is accepted as a natural extension of small mixing angles between quarks in the earlier period. Usually, one uses the mixing matrix written in a form of three submatrix product, each containing a mixing angle [4]. In recent years, the large vacuum mixing angle between ν_μ and ν_τ has been established almost certainly by the atmosphere neutrino experiments [5]. Large vacuum mixing angle between ν_e and ν_μ has been suggested also by some authors in combined research of solar neutrinos and the day night effect etc. [6]. Generally, the ν_e and ν_τ mixing angle is believed to be small [7]. With small mixing angles, the mixing order is not important. There are no difficulties in the explanation of their physical meanings. However, with large mixing, the product order become important. Therefor, it is possible that some important physical characteristic is hidden uncovered when the order of the product is not chosen in a good fashion. Some time it leads to misunderstanding.

A better general mathematical framework is desirable. We presented one in this paper using the direct physical parameters η_i. In present framework, the explicit analytical results are obtained. The mass eigenvalues, eigenfunctions, and mixing parameters in matter are represented analytically by vacuum mass, mixing parameters, and the effective potential A from charged current interaction. They are expressed in completely systematical form. There is no problem about mixing orders. And the physical connotations are exposed more clearly. It will be valuable as a powerful tool for further researches in three flavors neutrino oscillation in matter. We restrict ourselves to a most useful but special case, the three flavor mixing in matter with large mass differences, but without the small vacuum mixing angle restriction. For simplicity, the CP phase is ignored. There is no difficulty to add it in this theory. We discuss the charged current interaction between ν_e and e only.

II. MIXING MATRIXES BETWEEN THE EIGENVECTORS OF FLAVOR AND MASSES IN MATTER AND FLAVOR AND MASS IN VACUUM

The mixing matrix U^m which links the neutrino mass eigenstates in matter to the neutrino flavor eigenstates is defined as

$$\langle \nu_\alpha | = U^m_{\alpha u} \langle \nu^m_u | \tag{1}$$

where $|\nu_\alpha\rangle$, $\alpha = e, \mu, \tau$, and $|\nu^m_u\rangle$, $u = 1, 2, 3$, represent the three orthogonal and normalized eigenvectors of flavors and mass in matter of the neutrino and $\langle \nu_\alpha |$ and $\langle \nu^m_u |$ are their dual respectively. Summation convention for repeated indices has been used. We always use this rule below otherwise we indicate especially. Thus

$$U^m_{\alpha u} = \langle \nu_\alpha | \, \nu^m_u \rangle \tag{2}$$

If we use $|\nu\rangle$ to represent the state vector of a neutrino and ν_α $\alpha = e, \mu, \tau$ and ν^m_u $u = 1, 2, 3$ to express its components in the two bases spanned by $|\nu_\alpha\rangle$ and $|\nu^m_u\rangle$ respectively. That is

$$\nu_\alpha = \langle \nu_\alpha | \, \nu \rangle \tag{3}$$

$$\nu^m_u = \langle \nu^m_u | \, \nu \rangle \tag{4}$$

then, we have

$$\nu_\alpha = U^m_{\alpha u} \nu^m_u \tag{5}$$

However because $\langle \nu^m_u | \, \nu_\alpha \rangle = \langle \nu_\alpha | \, \nu^m_u \rangle$ is real in our Hilbert space, so we have also

$$|\nu_\alpha\rangle = U^m_{\alpha u} \, |\nu^m_u\rangle \tag{6}$$

In the same way, we define another transformation matrix U as

$$\langle \nu_\alpha | = U_{\alpha i} \langle \nu_i | \tag{7}$$

where $|\nu_i\rangle$ $i = 1, 2, 3$ represent the three orthogonal and normalized eigenvectors of mass in vacuum of the neutrino. We have

$$U_{\alpha i} = \langle \nu_\alpha | \, \nu_i \rangle \tag{8}$$

Let

$$\nu_i = \langle \nu_i | \, \nu \rangle \tag{9}$$

We have also

$$\nu_\alpha = U_{\alpha i} \nu_i \tag{10}$$

and

$$|\nu_\alpha\rangle = U_{\alpha i} \, |\nu_i\rangle \tag{11}$$

It should be noticed that we have used the $\alpha, \beta, ... = e, \mu, \tau$, the $u, v, ... = 1, 2, 3$ and the $i, j... = 1, 2, 3$ as the indices of the neutrino eigenvectors of flavor, mass in matter, and mass in vacuum respectively.

We would like to have a set of relations between the matrix elements $U^m_{\alpha u}$ and $U_{\alpha i}$, using the vacuum masses m_i and potential A as parameters, and $U^m_{\alpha u}$ are expressed in explicit analytical form. For this, we separate U^m into two factors

$$U^m = UW \tag{12}$$

Writing in component form

$$U^m_{\alpha u} = U_{\alpha i} W_{iu} \tag{13}$$

it is easy to see W is a transformation matrix which connect the eigenvectors of mass in vacuum and mass in matter. It can be expressed as

$$W_{iu} = \langle \nu_i | \, \nu^m_u \rangle \tag{14}$$

Now it is clear that our task is to search for the $|\nu^m_u\rangle$ represented in the base spanned by eigenvectors of mass in vacuum.

III. EIGENVALUES AND EIGENFUNCTOINS OF MASS MATRIX IN MATTER

Writing in flavor representation, the mass matrix in matter of three generation neutrinos is [4]:

$$M_f^2 = U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

Where m_i, $i = 1, 2, 3$, is the neutrino vacuum masses. A is an efficient potential of the electron neutrino ν_e in matter. As usual, we consider only charged current interaction between ν_e and the electron e in matter.

$$A = 2\sqrt{2} G_f N_e E \tag{16}$$

Where G_f is the Fermi constant, N_e is the electron number density of the matter, and E is the energy of the neutrino.

Our aim is to search for the $W_{iu} = \langle \nu_i | \, \nu_u^m \rangle$. So it is necessary to write the matrix of mass in matter in the vacuum mass representation

$$.M_m^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + U^\dagger \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \tag{17}$$

Let

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \end{pmatrix} \tag{18}$$

That is $\eta_i = U_{ei}$, $i = 1, 2, 3$. It is easy to show

$$U^\dagger \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U = A\eta^\dagger \eta = A \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \tag{19}$$

where $\eta_{ij} = \eta_i \eta_j = \eta_{ji}$. We do not accept the traditional mixing angles, but instead of the use of original physical parameters $\eta_i = U_{\alpha i} = \langle \nu_\alpha | \, \nu_i \rangle$ in our theoretical framework. By this, all the expressions are simple, the results are expressed in symmetrical form, and the physical contents are clear, not only for the mixing matrix discussed in this paper but also for the expressions of resonance, propagation equation, and so on, discussed in followed papers. There is a relation between three η_i. It is easy to prove that

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = 1 \tag{20}$$

Taking the trace to η_{ij} matrix, we obtain

$$\mathrm{Tr}\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \equiv \sum_{i=1}^{3} \eta_i^2 = \mathrm{Tr}\left[U^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \right] = 1 \tag{21}$$

Thus, we have two independent parameters, not three. We can go further to simplify the Eq.(17) as usual by subtracting the same mass $(m_1^2 + m_2^2)/2$ from the M_m^2. The mass matrix in matter can be written as

$$M_m^2 = \frac{1}{2}\left(m_2^2 + m_1^2\right) + \overline{M}_m^2 \tag{22}$$

Where

$$\overline{M}_m^2 = \begin{pmatrix} -\Delta m_1^2 & 0 & 0 \\ 0 & \Delta m_1^2 & 0 \\ 0 & 0 & \Delta m_2^2 \end{pmatrix} + A \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \tag{23}$$

and

$$\Delta m_1^2 = \frac{1}{2}\left(m_2^2 - m_1^2\right), \quad \Delta m_2^2 = m_3^2 - \frac{1}{2}\left(m_2^2 + m_1^2\right) \tag{24}$$

The eigenvalue equation of $\overline{M}_m^2 = M_m^2 - (m_1^2 + m_2^2)/2$ becomes

$$\overline{M}_m^2 \begin{pmatrix} \nu_1^m \\ \nu_2^m \\ \nu_3^m \end{pmatrix} = \lambda \begin{pmatrix} \nu_1^m \\ \nu_2^m \\ \nu_3^m \end{pmatrix} \tag{25}$$

Then, the dependence on the three vacuum masses m_i^2, $(i = 1, 2, 3,)$ is reduced into two Δm_i^2, $(i = 1, 2)$. The condition for a nontrivial solution to be existence for the eigenstate equation is

$$\det \begin{pmatrix} -\Delta m_1^2 + A\eta_{11} - \lambda & A\eta_{12} & A\eta_{13} \\ A\eta_{21} & \Delta m_1^2 + A\eta_{22} - \lambda & A\eta_{23} \\ A\eta_{31} & A\eta_{32} & \Delta m_2^2 + A\eta_{33} - \lambda \end{pmatrix} = 0 \tag{26}$$

Expanding it, we get a algebraic equation in third order.

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{27}$$

where

$$a = -(A + \Delta m_2^2) \tag{28}$$

$$b = -(\Delta m_1^2)^2 + \left[\Delta m_1^2\left(\eta_{11} - \eta_{22}\right) + \Delta m_2^2\left(\eta_{11} + \eta_{22}\right)\right] A \tag{29}$$

and

$$c = (\Delta m_1^2)^2 \Delta m_2^2 - \Delta m_1^2 \left[\Delta m_2^2\left(\eta_{11} - \eta_{22}\right) - \Delta m_1^2 \eta_{33}\right] A \tag{30}$$

The solutions of λ are standard in algebra but somewhat complicated. For completeness we write them at below but will use it seldom. The three eigenvalues of mass in matter, M_u^2 ($u = 1, 2, 3,$), of the three generation neutrinos are:

$$M_{m,u}^2 = \lambda_u + \frac{1}{2}\left(m_1^2 + m_2^2\right) \qquad u = 1, 2, 3, \tag{31}$$

where

$$\lambda_u = Y_u - \frac{a}{3}, \qquad u = 1, 2, 3, \tag{32}$$

$$Y_1 = \beta_1 + \beta_2, \quad Y_2 = \alpha_1\beta_1 + \alpha_2\beta_2, \quad and \quad Y_3 = \alpha_1\beta_2 + \alpha_2\beta_1 \tag{33}$$

$$\alpha_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \alpha_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \tag{34}$$

$$\beta_1 = \left(\sqrt{Q} - \frac{q}{2}\right)^{\frac{1}{3}}, \quad and \quad \beta_2 = \left(-\sqrt{Q} - \frac{q}{2}\right)^{\frac{1}{3}} \tag{35}$$

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{36}$$

$$p = b - \frac{1}{3}a^2, \quad q = \frac{2}{27}a^3 - \frac{1}{3}ab + c \tag{37}$$

Having the eigenvalues solved, the eigenfunctions of mass in matter, represented in the frame spanned by the vacuum mass eigenvectors, can be obtained by solving the eigenstate equation, Eq.(25). Rewriting Eq.(25) in a form as follows

$$A\begin{pmatrix}\eta_{11} & \eta_{12} & \eta_{13}\\ \eta_{21} & \eta_{22} & \eta_{23}\\ \eta_{31} & \eta_{32} & \eta_{33}\end{pmatrix}\begin{pmatrix}\nu_1^u\\ \nu_2^u\\ \nu_3^u\end{pmatrix}=\begin{pmatrix}\lambda_u+\Delta m_1^2 & 0 & 0\\ 0 & \lambda_u-\Delta m_1^2 & 0\\ 0 & 0 & \lambda_u-\Delta m_2^2\end{pmatrix}\begin{pmatrix}\nu_1^u\\ \nu_2^u\\ \nu_3^u\end{pmatrix} \tag{38}$$

After routine algebra, we can obtain the solution.

$$\nu_i^{m,u}=\langle\nu_i\mid\nu_u^m\rangle=N^u\frac{\eta_i}{\lambda_u-\Delta_i}\qquad u=1,2,3,\quad i=1,2,3 \tag{39}$$

where

$$\Delta_1=-\Delta m_1^2,\quad \Delta_2=\Delta m_1^2,\quad \Delta_3=\Delta m_2^2\qquad i=1,2,3 \tag{40}$$

and $\mid\nu^{m,u}\rangle$ is the eigenvector corresponding the mass $M_{m,u}$, $u=1,2,3$, and N^u is a normalized constant. It is given by

$$N^u=\left[\sum_{i=1}^{3}\left(\frac{\eta_i}{\lambda_u-\Delta_i}\right)^2\right]^{-\frac{1}{2}}\qquad u=1,2,3 \tag{41}$$

In the above three equations, the repeated index i does not subject to the summing rule. Because $\lambda_u-\Delta_i=M_{m,u}^2-m_i^2$, Eq.(39) and Eq.(41) can also be written in a form using original parameters $M_{m,u}^2$ and m_i.

$$\nu_i^{m,u}=\langle\nu_i\mid\nu_u^m\rangle=N^u\frac{\eta_i}{M_{m,u}^2-m_i^2}\qquad u=1,2,3,\quad i=1,2,3 \tag{42}$$

$$N^u=\left[\sum_{i=1}^{3}\left(\frac{\eta_i}{M_{m,u}^2-m_i^2}\right)^2\right]^{-\frac{1}{2}}\qquad u=1,2,3, \tag{43}$$

However, writing in this form, we need three vacuum mass m_i, but in the Eq.(39) and Eq.(41), it is only necessary to have two vacuum mass differences known.

IV. THE EXPRESSION OF MIXING MATRIX IN MATTER AND THE PROBLEM OF MIXING ORDER

By Eq.(13) and Eq.(14), we have $U_{\alpha u}^m=U_{\alpha i}W_{iu}$ and $W_{iu}=\langle\nu_i|\,\nu_u^m\rangle=N^u\eta_i/\left(\lambda_u-\Delta_i\right)=N^u\eta_i/\left(M_u^2-m_i^2\right)$. Then we have

$$U_{\alpha u}^m=N^u\sum_{i=1}^{3}U_{\alpha i}\frac{\eta_i}{\lambda_u-\Delta_i}=N^u\sum_{i=1}^{3}U_{\alpha i}\frac{\eta_i}{M_u^2-m_i^2}\quad \alpha=e,\mu,\tau,\ u=1,2,3 \tag{44}$$

In particular, we have the very simple form for ν_e mixing

$$U_{eu}^m = N^u \sum_{i=1}^{3} \frac{\eta_i^2}{\lambda_u - \Delta_i} = N^u \sum_{i=1}^{3} \frac{\eta_i^2}{M_u^2 - m_i^2} \qquad u = 1, 2, 3 \tag{45}$$

In the traditional theory, one accepts the form of a matrix with three mixing angles [4]

$$U = U_2 U_3 U_1 \tag{46}$$

where

$$U_1 = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{47}$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \tag{48}$$

$$U_3 = \begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix} \tag{49}$$

in which $c_i = \cos\theta_i$ and $s_i = \sin\theta_i,\ i = 1, 2, 3$. θ_i are called vacuum mixing angles. When all of the θ_i are small, the orders of the three rotations are not important. From mathematical and physical view, expressions in different orders are equivalent when the terms with higher order small quantities can be neglected. In physics, they represent three successive rotation angles in three different planes respectively. When any of the mixing angle θ_i is large, we can not explain as that, at least to some of the angles. In different orders, each θ_i represents a different physical content. We call it the problem of orders. In the same way, mixing in matter is also expressed as a product of three matrix in traditional theory

$$U^m = U_2^m U_3^m U_1^m \tag{50}$$

where U_i^m can be obtained using Eq.(47)-Eq.(49) by replacing the c_i and s_i by c_i^m and s_i^m respectively and let $c_i^m = \cos\theta_i^m$ and $s_i^m = \sin\theta_i^m,\ i = 1, 2, 3$. The θ_i^m is called mixing angles in matter. The problem of mixing order is the same as that in vacuum mixing case.

However, in our expressions, there is no order problem at all. The order problem is more subtle in the resonance phenomena in matter. We will discuss it in more details in a followed paper. In there, the advantage of our present theoretical framework will be more remarkable.

V. CONCLUSION

We have given a set of explicit analytical expressions connecting mixing matrix of three neutrinos in matter and vacuum parameters. In these formulae we have used a set of new physical parameters directly, instead of those used in traditional theory. Our results are simple, symmetrical, and physically clear. Moreover, there is no order problem here.

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